1 Abstract Scattering Theory

1.1 Time and energy picture

Scattering theory is mostly introduced in the *time picture* where one has to deal with differential equations and boundary conditions. In the *energy picture*, these equations become simple *algebraic equations* for operators. The connection between time and energy picture is given by the Fourier transform

$$f(t) = \frac{1}{2\pi\hbar}\int dE\, e^{-\frac{i}{\hbar}Et}\tilde{f}(E) \tag{1}$$

$$\tilde{f}(E) = \int dt\, e^{\frac{i}{\hbar}Et} f(t). \tag{2}$$

The main actor in the time picture is the *time evolution operator* or *propagator*

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}, \tag{3}$$

because the time evolution of any state, who is at time $t = 0$ in the state $|\psi\rangle$, reads

$$|\psi(t)\rangle = \hat{U}(t)|\psi\rangle. \tag{4}$$

Hence, all we have to know is the explicit form of $\hat{U}(t)$. In the energy picture, we have to look for the Fourier transform of $\hat{U}(t)$. However, the operator $\hat{U}(t)$ cannot be Fourier transformed, because it does not "converge". In order to accomplish a transformation, we introduce *retarded* and *advanced* propagators

$$U_{\pm}(t) := \pm\theta(\pm t)\, e^{-\frac{i}{\hbar}Ht}\, e^{\mp\epsilon t}, \tag{5}$$

where θ is the Heaviside step function and $\epsilon \to 0$. The retarded propagator cares for the time evolution towards the future, while the advanced propagator evolves for the past. In most situations we are only interested in the *future* evolution, so the retarded propagator,

$$\hat{U}_{+}(t) = \theta(t)e^{-\frac{i}{\hbar}(\hat{H}-i\epsilon)t} \tag{6}$$

is more useful to us. Anyway, we decompose $\hat{U}(t)$ along

$$\hat{U}(t) = \hat{U}_{+}(t) + \hat{U}_{-}(t), \tag{7}$$

and try to Fourier transform both parts separately. Let $|E'\rangle$ be an eigenvector of $\hat{H}$, then

$$\int dt\, e^{\frac{i}{\hbar}Et}\hat{U}_{+}(t)|E'\rangle = \int_0^{\infty} dt\, e^{-\frac{i}{\hbar}(E'-E-i\epsilon)t}|E'\rangle \tag{8}$$

$$= \Big[\frac{\hbar}{-i(E'-E-i\epsilon)}e^{-\frac{i}{\hbar}(E'-E-i\epsilon)t}\Big]_{t=0}^{\infty} \tag{9}$$

$$= i\hbar\frac{1}{E-E'+i\epsilon}|E'\rangle. \tag{10}$$

Analoguously, we have

$$\int dt\, e^{\frac{i}{\hbar}Et}\hat{U}_-(t)|E'\rangle = -i\hbar\frac{1}{E-E'-i\epsilon}|E'\rangle. \tag{11}$$

Since the $|E'\rangle$ form a complete basis we have

$$\int dt\, e^{\frac{i}{\hbar}Et}\hat{U}_\pm(t) = \pm i\hbar\,\hat{R}_\pm(E), \tag{12}$$

or

$$\int dt\, e^{\frac{i}{\hbar}Et}\hat{U}(t) = i\hbar\big\{\hat{R}_+(E) - \hat{R}_-(E)\big\}, \tag{13}$$

where

$$\hat{R}_\pm(E) := \frac{1}{E-\hat{H}\pm i\epsilon} \tag{14}$$

is the retarded and advanced *Green operator*, respectively. The Green operators are the *analytic continuations* of the so-called *resolvent* of $\hat{H}$,

$$\hat{R}(z) = \frac{1}{z-\hat{H}}, \tag{15}$$

with $\hat{R}_\pm(E) = \hat{R}(E\pm i\epsilon)$. Conversely, we have

$$\hat{U}(t) = \frac{i}{2\pi}\int dE\, e^{-\frac{i}{\hbar}Et}\big\{\hat{R}_+(E) - \hat{R}_-(E)\big\}, \tag{16}$$

which can be calculated using the residue theorem. So the Green operator is the Fourier transform of the time evolution operator, such that we can simply switch between time and energy picture.

1.2 Spectral decomposition using the resolvent

Since the resolvent is analytic except on the spectrum of $\hat{H}$, both analytic continuations $R_\pm(E)$ coincide here, $R_+(E) = R_-(E)$, so the integral vanishes everywhere except on the spectrum σ. Assume that $|E'\rangle$ is an eigenvector to the (discrete or continuous) eigenvalue E'. Then

$$R_\pm(E)|E'\rangle = \frac{1}{E-E'\pm i\epsilon}|E'\rangle \tag{17}$$

$$= \left\{\frac{\mathcal{P}_{E'}}{E-E'} \mp i\pi\delta(E-E')\right\}|E'\rangle, \tag{18}$$

where $\mathcal{P}_{E'}$ is the *principal value* distribution, defined by

$$\int dx\, \mathcal{P}_{x_0} f(x) := \int_{-\infty}^{x_0-\epsilon} dx\, f(x) + \int_{x_0+\epsilon}^{\infty} dx\, f(x). \tag{19}$$

Hence, we have

$$\Big\{\hat{R}_+(E) - \hat{R}_-(E)\Big\}|E'\rangle = -2\pi i\delta(E-E')|E'\rangle. \tag{20}$$

Distinguishing between discrete and continuous eigenvalues, we arrive at the representation

$$\frac{i}{2\pi}\Big\{R_+(E) - R_-(E)\Big\} = \sum_n \delta(E - E_n)|E_n\rangle\langle E_n| + \rho(E)|E\rangle\langle E|, \tag{21}$$

where $\rho(E)$ is the *energy density* vanishing outside the continuous spectrum σ_c of $\hat{H}$,

$$\rho(E) = 0 \quad \forall E \notin \sigma_c. \tag{22}$$

The continuous energy eigenstates are normalized by

$$\langle E|E'\rangle = \frac{1}{\rho(E)}\delta(E - E'). \tag{23}$$

which finally gives us a spectral decomposition of unity expressed in terms of the resolvent,

$$\frac{i}{2\pi}\int dE\,\Big\{R_+(E) - R_-(E)\Big\} = \sum_n |E_n\rangle\langle E_n| + \int_{\sigma_c} dE\,\rho(E)|E\rangle\langle E| = \mathbb{1}. \tag{24}$$

Note that we have not considered degeneracies up to now.

1.3 Scattering theory

While in the time picture we have to solve differential equations with boundary conditions, in the energy picture we have to solve eigenvalue equations and do some operator algebra. Scattering theory looks *much* simpler in the energy picture, so we will go for it now.
In the energy picture, the aim is to find the *spectral decomposition* (eigenvalues and eigenvectors) of the *total Hamiltonian*

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{25}$$

where $\hat{H}_0$ is the *free Hamiltonian* and $\hat{V}$ is some *perturbation*, often parametrized by a perturbation parameter λ, such that $\hat{V} \mapsto \lambda\hat{V}$. Scattering theory is nothing but a special case of perturbation theory, with the free Hamiltonian $\hat{H}_0$ given by

$$\hat{H}_0 = \frac{\hbar^2}{2m}\hat{\boldsymbol{k}}^2, \tag{26}$$

and $\hat{V}$ being called the *scattering potential*. The spectral decomposition of the free Hamiltonian is known,

$$\hat{H}_0\,|E^0\rangle = E\,|E^0\rangle. \tag{27}$$

The eigenvector $|E^0\rangle$ is not unique because the energies are all degenerate,

$$E = \frac{\hbar^2}{2m}k^2, \tag{28}$$

i.e. $|E^0\rangle$ can be any of the vectors $|\boldsymbol{k}\rangle$ with $k := |\boldsymbol{k}| = \sqrt{2mE}/\hbar$. Hence, we switch to the momentum picture,

$$|E^0\rangle \mapsto |\boldsymbol{k}\rangle \tag{29}$$

$$\int dE\,\rho(E) \mapsto \int d^3k. \tag{30}$$

$$E \mapsto E_{\boldsymbol{k}} \equiv \frac{\hbar^2}{2m}k^2. \tag{31}$$

1.4 Bound states

Rewriting the eigenvalue equation of H_0,

$$\hat{H}_0 \, |\boldsymbol{k}\rangle = E_{\boldsymbol{k}} \, |\boldsymbol{k}\rangle \quad \Leftrightarrow \quad (E_{\boldsymbol{k}} - \hat{H}_0)|\boldsymbol{k}\rangle = |\emptyset\rangle, \tag{32}$$

where $|\emptyset\rangle$ is the null vector, we see that the eigenvalues $E_{\boldsymbol{k}}$ of $\hat{H}_0$ are just the points where the operator $(E_{\boldsymbol{k}} - \hat{H}_0)$ is not invertible. In other words: The eigenvalues are those points where the *free resolvent*,

$$\hat{R}_0(z) := \frac{1}{z - \hat{H}_0} \tag{33}$$

as a function of the complex number z, is not defined. Anywhere else, the resolvent is an operator-valued *analytic* function of z. The spectrum of H_0 is purely continuous and is given by the positive semi-axis $[0, \infty)$. This defines a *cut* in the complex plane (see Fig. **??**). Now

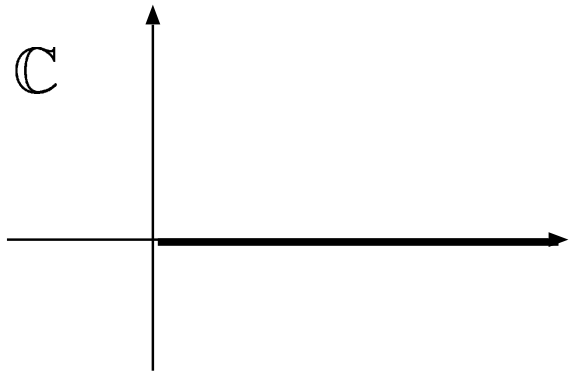

Figure 1: The spectrum of H_0, where the free resolvent $R_0(z)$ is not defined.

consider the *total resolvent*

$$\hat{R}(z) := \frac{1}{z - \hat{H}}. \tag{34}$$

Also here, the eigenvalues of the total Hamiltonian are those points where $\hat{R}(z)$ is not defined. There might be continuous eigenvalues as well as discrete eigenvalues. A mathematical theorem states that (under certain quite general conditions for $\hat{V}$) the continuous spectrum of $\hat{H}$ is not affected by $\hat{V}$. Thus, the continuous spectrum σ_c is still $[0, \infty)$. Any new discrete eigenvalues appearing *embedded* in the continuous spectrum are *instable* (discrete resonance states, only appearing on a countable set of perturbation strengths $\lambda_1, \lambda_2, \ldots$), so the *stable* discrete spectrum σ_d must be negative, corresponding to *bound states* (see Fig.**??**). Denoting the bound states with $|E_n\rangle$, we have

$$\hat{H}|E_n\rangle = E_n|E_n\rangle. \tag{35}$$

The bound states can be constructed with the help of the resolvent. The discrete eigenvalues E_n of $\hat{H}$ are the poles of the resolvent $\hat{R}(z)$. Let $|\phi_n\rangle$ be a Hilbert state such that the function

$$G_n(z) := \langle \phi_n | \hat{R}(z) | \phi_n \rangle \tag{36}$$

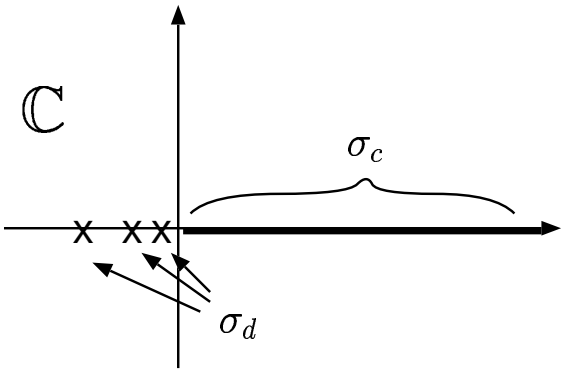

Figure 2: The spectrum of H, with bound states corresponding to negative discrete eigenvalues.

has a pole at $z = E_n$. Then the state

$$|E_n^{\pm}\rangle := \alpha \, \frac{\pm i\epsilon}{E_n - \hat{H} \pm i\epsilon} |\phi_n\rangle \tag{37}$$

is an eigenstate of $\hat{H}$ to the eigenvalue E_n, where α is a normalization constant and $\epsilon \to 0$. In case of *stable* bound states the pole is *isolated* and

$$|E_n^+\rangle = |E_n^-\rangle \equiv |E_n\rangle. \tag{38}$$

Let us only consider stable bound states in the following.

1.5 Møller operators

The resolvent can be used to construct eigenvectors. One can show that to each continuous eigenvalue $E_{\boldsymbol{k}}$ there are two different eigenvectors,

$$\hat{H}|E_\pm(\boldsymbol{k})\rangle = E_{\boldsymbol{k}}|E_\pm(\boldsymbol{k})\rangle, \tag{39}$$

called the *retarded* and the *advanced* solution, given by

$$\begin{aligned} |E_\pm(\boldsymbol{k})\rangle &= \pm i\epsilon R(E_{\boldsymbol{k}} \pm i\epsilon)|\boldsymbol{k}\rangle \\ &= \frac{\pm i\epsilon}{E_{\boldsymbol{k}} - \hat{H} \pm i\epsilon}|\boldsymbol{k}\rangle, \end{aligned} \tag{40}$$

with $\epsilon \to 0$ understood as usual. The retarded solution corresponds to an *outgoing* wave and is physically more comprehensive than the advanced solution, which correspond to an *incoming* wave. However, both solutions can equivalently (but not simultaneously!) be used in the spectral decomposition. Together with the bound states, we arrive at the spectral decomposition

$$\mathbb{1} = \underbrace{\sum_n |E_n\rangle\langle E_n|}_{\mathbb{1}_d} + \underbrace{\int d^3k \, |E_\pm(\boldsymbol{k})\rangle\langle E_\pm(\boldsymbol{k})|}_{\mathbb{1}_c}, \tag{41}$$

where $\mathbb{1}_d$ and $\mathbb{1}_c$ are unities on the Hilbert space spanned by the discrete and the continuous eigenstates of $\hat{H}$, thus

$$\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c. \tag{42}$$

The operators that map the free states $|\boldsymbol{k}\rangle$ onto the retarded or advanced continuous eigenstates $|E_\pm(\boldsymbol{k})\rangle$ are known as the *Møller operators*

$$\hat{\Omega}_\pm |\boldsymbol{k}\rangle := |E_\pm(\boldsymbol{k})\rangle. \tag{43}$$

Hence, the Møller operators read

$$\hat{\Omega}_\pm = \int d^3k \, |E_\pm(\boldsymbol{k})\rangle\langle\boldsymbol{k}|. \tag{44}$$

So the Møller operators are *almost unitary*:

$$\hat{\Omega}_\pm^\dagger \hat{\Omega}_\pm = \int d^3k \int d^3k' \, |\boldsymbol{k}\rangle\langle E_\pm(\boldsymbol{k})|E_\pm(\boldsymbol{k}')\rangle\langle\boldsymbol{k}'| \tag{45}$$

$$= \mathbb{1}, \tag{46}$$

but

$$\hat{\Omega}_\pm \hat{\Omega}_\pm^\dagger = \int d^3k \int d^3k' \, |E_\pm(\boldsymbol{k})\rangle\langle\boldsymbol{k}|\boldsymbol{k}'\rangle\langle E_\pm(\boldsymbol{k}')| \tag{47}$$

$$= \int d^3k \, |E_\pm(\boldsymbol{k})\rangle\langle E_\pm(\boldsymbol{k})| = \mathbb{1}_c, \tag{48}$$

which is only the unity on the subspace $\mathcal{H}_c$ spanned by the continuous eigenvectors. If there were no bound states, the Møller operators would be unitary.

1.6 Lippmann-Schwinger equation and Born approximation

From now on we restrict to retarded solutions. Equation (**??**) is not yet really helpful as it involves the operator $\hat{R}(z)$, whose spectral decomposition is unknown. Luckily, there is the very important *second resolvent identity*, stating

$$\hat{R}(z) = \hat{R}_0(z) + \hat{R}_0(z)\hat{V}\hat{R}(z). \tag{49}$$

If we insert (**??**) into (**??**), we get

$$|E_+(\boldsymbol{k})\rangle = i\epsilon\big\{\hat{R}_0(E_{\boldsymbol{k}} + i\epsilon) + \hat{R}_0(E_{\boldsymbol{k}} + i\epsilon)\hat{V}\hat{R}(E_{\boldsymbol{k}} + i\epsilon)\big\}|\boldsymbol{k}\rangle \tag{50}$$

$$= \frac{i\epsilon}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}|\boldsymbol{k}\rangle + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}\, i\epsilon\hat{R}(E_{\boldsymbol{k}} + i\epsilon)|\boldsymbol{k}\rangle \tag{51}$$

Using (**??**) this gives the *Lippmann-Schwinger-equation*

$$|E_+(\boldsymbol{k})\rangle = |\boldsymbol{k}\rangle + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}\,|E_+(\boldsymbol{k})\rangle. \tag{52}$$

In terms of Møller operators it reads

$$\hat{\Omega}_+|\boldsymbol{k}\rangle = \Big\{\mathbb{1} + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}\hat{\Omega}_+\Big\}|\boldsymbol{k}\rangle. \tag{53}$$

Note that since the operator on the righthand side also involves $\boldsymbol{k}$, one cannot generalize to the entire Hilbert space, i.e.

$$\hat{\Omega}_+ \neq \mathbb{1} + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}\hat{\Omega}_+. \tag{54}$$

However, one can use (**??**) without getting into trouble. Now we insert the unity decomposed in $|\boldsymbol{k}\rangle$, yielding

$$|E_+(\boldsymbol{k})\rangle = |\boldsymbol{k}\rangle + \int d^3k' \frac{1}{E_{\boldsymbol{k}} - E_{\boldsymbol{k}'} + i\epsilon}\langle \boldsymbol{k}'|\hat{V}|E_+(\boldsymbol{k})\rangle\,|\boldsymbol{k}'\rangle. \tag{55}$$

Thus, in position representation we have

$$\sqrt{2\pi}^3\langle \boldsymbol{x}|E_+(\boldsymbol{k})\rangle = e^{i\boldsymbol{k}\boldsymbol{x}} + \int d^3k' \frac{e^{i\boldsymbol{k}'\boldsymbol{x}}}{E_{\boldsymbol{k}} - E_{\boldsymbol{k}'} + i\epsilon}\langle \boldsymbol{k}'|\hat{V}|E_+(\boldsymbol{k})\rangle. \tag{56}$$

The term $\langle \boldsymbol{k}'|\hat{V}|E_+(\boldsymbol{k})\rangle$ is known as the *scattering amplitude*:

$$f(\boldsymbol{k},\boldsymbol{k}') := -\frac{4\pi^2 m}{\hbar^2}\langle \boldsymbol{k}'|\hat{V}|E_+(\boldsymbol{k})\rangle, \tag{57}$$

and its modulus square as the *differential cross section*:

$$\frac{d\sigma}{d\Omega} := |f(\boldsymbol{k},\boldsymbol{k}')|^2. \tag{58}$$

Equation (**??**) does not help very much, because it is an implicit equation. However, it can be iterated, giving the *Born series*

$$|E_+(\boldsymbol{k})\rangle = \sum_{j=0}^{\infty}\Big\{\frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}\Big\}^j|\boldsymbol{k}\rangle. \tag{59}$$

As a series in powers of $\hat{V}$, they are *perturbation expansions*. Cutting the series in first order gives the *first Born approximation*

$$|E_+(\boldsymbol{k})\rangle \approx |\boldsymbol{k}\rangle + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{V}|\boldsymbol{k}\rangle \tag{60}$$

for the energy eigenstate. Thus, in first Born approximation we have for the scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}') \approx -\frac{4\pi^2 m}{\hbar^2}\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle, \tag{61}$$

and for the approximate eigenstate

$$|E_+(\boldsymbol{k})\rangle \approx |\boldsymbol{k}\rangle + \int d^3k' \frac{1}{\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k'^2}{2m} + i\epsilon} \langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle \, |\boldsymbol{k}'\rangle \tag{62}$$

Changing to polar coordinates and considering the pole at $k = k' + i\epsilon$ in the upper half-plane, we can integrate the above expression using the residue theorem, which gives

$$E_{\boldsymbol{k}}(\boldsymbol{x}) \approx e^{i\boldsymbol{k}\boldsymbol{x}} + \frac{e^{ikx}}{x} f(\boldsymbol{k}', \boldsymbol{k}), \tag{63}$$

where $E_{\boldsymbol{k}}(\boldsymbol{x}) := \sqrt{2\pi}^3 \langle \boldsymbol{x}|E_+(\boldsymbol{k})\rangle$. So the scattered wave is a superposition of the free wave $e^{i\boldsymbol{k}\boldsymbol{x}}$ and an *outgoing* spherical wave e^{ikx} decreasing with $f(\boldsymbol{k}', \boldsymbol{k})/x$. Just in order to illustrate, take the *advanced* solution,

$$|E_-(\boldsymbol{k})\rangle \approx |\boldsymbol{k}\rangle + \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 - i\epsilon} \hat{V}|\boldsymbol{k}\rangle. \tag{64}$$

The pole is now in the *upper* half-plane at $k = k' - i\epsilon$, which would lead to the amplitude

$$E_{\boldsymbol{k}}(\boldsymbol{x}) \approx e^{i\boldsymbol{k}\boldsymbol{x}} - \frac{e^{-ikx}}{x} f(\boldsymbol{k}', \boldsymbol{k}), \tag{65}$$

representing a free wave plus an *incoming* spherical wave e^{-ikx}, which is physical nonsense.

1.7 The T-Matrix

The scattering amplitude (**??**) can be regarded as the matrix element of the so-called *Transfer matrix* or *T-Matrix*:

$$f(\boldsymbol{k}', \boldsymbol{k}) =: -\frac{4\pi^2 m}{\hbar^2} \langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle, \tag{66}$$

or, by comparing with (**??**),

$$\hat{T}|\boldsymbol{k}\rangle := \hat{V}|E_+(\boldsymbol{k})\rangle, \tag{67}$$

or, in terms of Møller operators,

$$\hat{T} := \hat{V}\hat{\Omega}_+. \tag{68}$$

Multiplying (**??**) from the left with $\hat{V}$, we arrive at the *Lippmann-Schwinger-equation for the T-Matrix*.

$$\hat{T}|\boldsymbol{k}\rangle = \left\{\hat{V} + \hat{V}\frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{T}\right\}|\boldsymbol{k}\rangle. \tag{69}$$

Unfortunately, the operator on the righthand side involves $\boldsymbol{k}$, so one *cannot* generalize $\hat{T}$ to the entire Hilbert space,

$$\hat{T} \neq \hat{V} + \hat{V}\frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon}\hat{T}. \tag{70}$$

Rather, one should use (**??**) instead, whose corresponding Born series reads

$$\hat{T}|\boldsymbol{k}\rangle = \sum_{j=0}^{\infty} \Big\{ \hat{V} \frac{1}{E_{\boldsymbol{k}} - \hat{H}_0 + i\epsilon} \Big\}^j \hat{V}|\boldsymbol{k}\rangle. \tag{71}$$

So in first Born approximation the T-Matrix simply coincides with the scattering potential,

$$\hat{T} \approx \hat{V}. \tag{72}$$

1.8 The S-Matrix

The retarded and advanced eigenstates $|\psi_\pm(\boldsymbol{k})\rangle$ can equivalently be used in the unity decomposition, so they cannot be linearly independent. Indeed, we have

$$\begin{aligned} |E_+(\boldsymbol{k})\rangle &= \{\mathbb{1}_d + \mathbb{1}_c\}|E_+(\boldsymbol{k})\rangle = \mathbb{1}_c\,|E_+(\boldsymbol{k})\rangle && (73)\\ &= \int d^3k'\,|E_-(\boldsymbol{k}')\rangle\langle E_-(\boldsymbol{k}')|E_+(\boldsymbol{k})\rangle && (74)\\ &= \int d^3k\,S(\boldsymbol{k}',\boldsymbol{k})|E_-(\boldsymbol{k}')\rangle, && (75) \end{aligned}$$

where

$$S(\boldsymbol{k}',\boldsymbol{k}) = \langle \boldsymbol{k}'|\hat{S}|\boldsymbol{k}\rangle = \langle E_-(\boldsymbol{k}')|E_+(\boldsymbol{k})\rangle \tag{76}$$

are the matrix elements of the so-called *scattering matrix* or *S-Matrix*. In momentum decomposition, the S-Matrix reads

$$\begin{aligned} \hat{S} &= \int d^3k \int d^3k' S(\boldsymbol{k}',\boldsymbol{k})|\boldsymbol{k}'\rangle\langle\boldsymbol{k}| && (77)\\ &= \int d^3k \int d^3k' \langle E_-(\boldsymbol{k}')|E_+(\boldsymbol{k})\rangle|\boldsymbol{k}'\rangle\langle\boldsymbol{k}| && (78)\\ &= \int d^3k \int d^3k' \langle \boldsymbol{k}'|\hat{\Omega}_-^\dagger \hat{\Omega}_+|\boldsymbol{k}\rangle|\boldsymbol{k}'\rangle\langle\boldsymbol{k}|, && (79) \end{aligned}$$

hence

$$\hat{S} = \hat{\Omega}_-^\dagger \hat{\Omega}_+. \tag{80}$$

So what is the use of the S-Matrix? Consider some *incoming* state $|\psi_{\text{in}}\rangle$ with the wave function $\psi_{\text{in}}(\boldsymbol{k}) := \langle \boldsymbol{k}|\psi_{\text{in}}\rangle$. Now we apply the S-Matrix to it and find

$$\begin{aligned} \hat{S}|\psi_{\text{in}}\rangle &= \int d^3k \int d^3k'\, S(\boldsymbol{k}',\boldsymbol{k})\psi_{\text{in}}(\boldsymbol{k})|\boldsymbol{k}'\rangle && (81)\\ &= \int d^3k' \psi_{\text{out}}(\boldsymbol{k}')|\boldsymbol{k}'\rangle, && (82) \end{aligned}$$

where

$$\psi_{\text{out}}(\boldsymbol{k}') := \int d^3k\, S(\boldsymbol{k}',\boldsymbol{k})\psi_{\text{in}}(\boldsymbol{k}), \tag{83}$$

is the wave function of some other state, which we call the *outgoing state*. Hence, the S-Matrix maps incoming states to the outgoing states:

$$\hat{S}|\psi_{\text{in}}\rangle = |\psi_{\text{out}}\rangle. \tag{84}$$

Now we can invent any names we want, the physical picture behind is the following. In the far past, the particle is a plane wave $|\psi_{\text{in}}\rangle = |\boldsymbol{k}\rangle$, which is an eigenstate of $\hat{H}_0$ with energy $E_{\boldsymbol{k}}$. At time $t = 0$, the particle enters a small interaction region, the realm of the scattering potential $\hat{V}$. Here it transforms into an eigenstate $|E_+(\boldsymbol{k})\rangle$ of the full Hamiltonian $\hat{H}$ to the same energy $E_{\boldsymbol{k}}$. The particle leaves the interaction region and in the far future it will be in a free state $|\psi_{\text{out}}\rangle = |\boldsymbol{k}'\rangle$ with $|\boldsymbol{k}'| = |\boldsymbol{k}|$, which is another eigenstate of $\hat{H}_0$ of still the same energy $E_{\boldsymbol{k}'} = E_{\boldsymbol{k}}$. The latter transition is the time reversed version of the former, hence here we must deal with the *advanced* eigenvector $|E_-(\boldsymbol{k})\rangle$. The total process reads

$$|\boldsymbol{k}\rangle \to |E_+(\boldsymbol{k})\rangle \to |E_-(\boldsymbol{k}')\rangle \to |\boldsymbol{k}'\rangle \tag{85}$$

and is depicted in Fig. **??**. By the superposition principle we can compose these plane waves into a wave package, which then represents a truly *physical* state. The physical interpretation of the scattering process, though, remains the same. In short: The matrix elements $S(\boldsymbol{k}', \boldsymbol{k})$

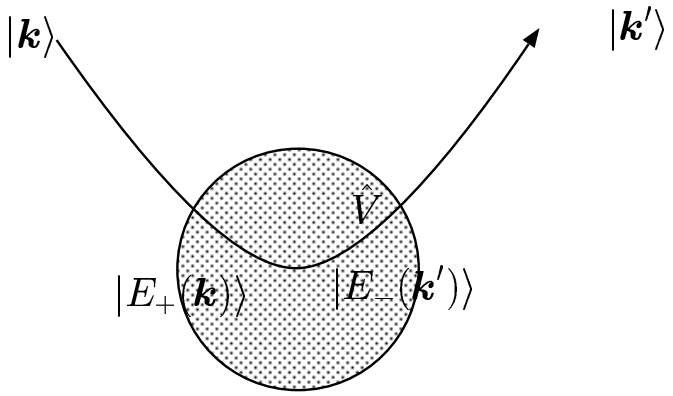

Figure 3: The scattering process.

give the probability amplitude that an *incoming state* $|\boldsymbol{k}\rangle$ is scattered onto an *outgoing state* $|\boldsymbol{k}'\rangle$:

$$|\boldsymbol{k}\rangle \overset{S(\boldsymbol{k}',\boldsymbol{k})}{\longrightarrow} |\boldsymbol{k}'\rangle. \tag{86}$$